論　説

On the Unique Existence of the Nash Equilibrium in Cournot Mixed Oligopoly with Linear Demand and Quadratic Cost Functions[+]

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Abstract

Okuguchi and Yamazaki (2017) prove that if each firm's marginal cost of its first unit of production is small enough, a public firm and at least one private firm produce the good in the unique Nash equilibrium of Cournot mixed oligopoly. In this paper, we study a concrete model of the mixed oligopoly where firms with a linear or quadratic cost function face a linear market demand function to show how small each firm's marginal cost of its first unit of production should be.

JEL Classification Numbers: L13, D43, H32.
Key Words: mixed oligopoly, Cournot, equilibrium, existence.

1 Introduction

Many researchers have studied Cournot mixed oligopoly in which one social welfare maximizing public firm and several profit-maximizing private firms compete one another in a single market. Almost all recent papers on Cournot mixed oligopoly have derived their results basing on numerical calculations of the equilibrium values for simple cases of a linear market demand function, and linear or quadratic cost functions. Okuguchi (1985) studies a general model of oligopoly with a competitive fringe, which can be interpreted as a general model of mixed oligopoly with one public firm and several private firms. Matsumura (1998) studies the effect of privatization of a public firm in a general model of mixed duopoly. Myles (2002) and Matsumura and Kanda (2005) study a general model of mixed oligopoly with one public firm and several private firms. However, to derive their interesting results, they analyze the equilibrium condition without proving the existence of a Nash equilibrium. Okuguchi (2012) proves the unique existence of the Nash equilibrium in a general model of Cournot mixed oligopoly, but one public firm or all private firms may not produce in the equilibrium. Okuguchi and Yamazaki (2017) formulate a general model

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of Cournot mixed oligopoly with one public firm and several private firms, and derive a set of general conditions ensuring the existence of a unique Nash equilibrium for the mixed oligopoly where one public firm and at least one of private firms are active in the equilibrium. They also explain that if each firm's marginal cost of its first unit of production is small enough, the set of conditions are satisfied so that the public firm and at least one private firm produce the good in the unique Nash equilibrium. In this paper, we will show how small each firm's marginal cost of its first unit of production should be in a concrete model of the mixed oligopoly where firms with a linear or quadratic cost function face a linear market demand function.

2 Model

As in the model of Okuguchi and Yamazaki (2017), let firm 0 be a public firm and firm i, $i=1,2,\cdots,n$, private one. All firms produce a homogeneous good to sell in a single market. Denote firm i's output as x_i, $i=0,1,\cdots,n$. Okuguchi and Yamazaki (2017) use the following notations.

> x_i, firm i's output, $i=0,1,\cdots,n$,
> $C_i(x_i)$, firm i's cost function, $i=0,1,\cdots,n$,
> $X\equiv\sum_{i=0}^{n}x_i$, industry output,
> $p=f(X)$, inverse market demand function, where p is the market price of the goods produced by all firms.

Okuguchi and Yamazaki (2017) impose the following assumptions.[1]

Assumption 1: *$C_i(x_i)$ is twice continuously differentiable and $C_i'(x_i)>0$ for any $x_i>0$, $i=0,1,\cdots,n$.*

Assumption 2: *$f(X)$ is twice continuously differentiable for any $X\in\mathbb{R}_{++}\setminus\{\bar{X}\}$, $f'(X)<0$ for any $X\in(0,\bar{X})$ and $f'(X)=0$ for any $X\geq\bar{X}$, where $\bar{X}\equiv\min\{X|f(X)=0\}$.*[2]

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[1] Okuguchi (1985, 2012) and Matsumura (1998) impose Assumptions 4 and 5 on private firms' cost functions in the mixed duopoly and oligopoly, respectively. Matsumura and Kanda (2005) impose an assumption that $C_i(x)=C(x)$ and $C''(x)>0$ for any $x>0$, $i=1,2,\cdots,n$. Okuguchi (1985) assumes $C_0''(x_0)>0$ for a competitive fringe, which can be interpreted as a public firm.

[2] If $f(X)>0$ for any $X>0$, we define $\bar{X}\equiv\infty$.

Assumption 3: $f(0) > \max\left\{C_i'(0)\right\}_{i=0}^{n}$, where $C_i'(0) \equiv \lim_{x_i \to 0+} C_i(x_i)$.[3]

Assumption 4: $f'(X) < C_i''(x_i)$ *for any* $x_i > 0$ *and* $X \in \left[x_i, \bar{X}\right)$, $i = 0,1,\cdots,n$.

Assumption 5: $f'(X) + xf''(X) \leq 0$ *for any* $x > 0$ *and* $X \geq x$.

Assumption 6: $f(x) < C_i'(x)$ *for some* $x > 0$, $i = 0,1,\cdots,n$.

In this paper, we assume that the inverse demand function has a specific form as follow.

$$f(X) = \begin{cases} a - bX & \text{for any } X \in \left[0, \bar{X}\right] \\ 0 & \text{for any } X > \bar{X} \end{cases}, \quad a > 0, \ b > 0, \tag{1}$$

where

$$\bar{X} \equiv a/b. \tag{2}$$

Furthermore, we assume that the cost functions of all firms are quadratic:

$$C_i(x_i) = c_0^i + c_1^i x_i + c_2^i (x_i)^2, \ c_0^i \geq 0, \ c_1^i > 0, \ c_2^i \geq 0, \ i = 1,2,\cdots,n. \tag{3}$$

Note that $C_i'(0) = c_1^i$, $i = 0,1,\cdots,n$. If $c_2^i = 0$, firm i's cost function is linear and $C_i'(x_i) = c_1^i$ for any $x_i > 0$, $i = 0,1,\cdots,n$. Without loss of generality, assume

$$c_1^n \geq c_1^{n-1} \geq \cdots \geq c_1^1 \geq 0. \tag{4}$$

Since $C_i'(x_i) \geq c_1^i > 0$ and $C_i''(x_i) = 2c_2^i \geq 0$ for any $x_i > 0$, $i = 0,1,\cdots,n$, Assumption 1 is satisfied. It is clear that Assumption 2 is satisfied with $\bar{X}$ in (2). In the following analysis, we assume

$$a > c_1^n = \max\left\{c_1^i\right\}_{i=0}^{n}, \tag{5}$$

which is equivalent to Assumption 3 in the model of this paper. Since $f'(X) \leq 0$ and $f''(X) = 0$ in the inverse demand function (1), Assumption 5 is clearly satisfied. Since $f'(X) = -b < 0$ and $C_i''(x_i) = 2c_2^i \geq 0$ for any $x_i > 0$ and $X \in \left[x_i, \bar{X}\right)$, $i = 0,1,\cdots,n$, Assumption 4 is also satisfied. Note that $f(X) - c_1^i < 0$ for any $X > \bar{X}^i$, where

$$\bar{X}^i \equiv \frac{1}{b}\left(a - c_1^i\right), \ i = 0,1,\cdots,n. \tag{6}$$

Since $C_i'(x) \geq c_1^i > 0$ for any $x > 0$ and $f(x) - c_1^i < 0$ for any $x > \bar{X}^i$, $f(x) < C_i'(x)$ for any $x > \bar{X}^i$. Hence, Assumption 6 is satisfied in the model of this paper.

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[3] As $C_i'(0) \equiv \lim_{x_i \to 0+} C_i(x_i)$, we define $f'(0) \equiv \lim_{X \to 0+} f'(X)$. By Assumption 1, $C_i'(0) \geq 0$. By Assumption 2, $f'(0) \leq 0$.

3 Equilibrium Analyses

The social welfare W is the sum of the consumers' surplus and profits of all firms.

$$W \equiv \int_0^X f(x)dx - \sum_{i=0}^{n} C_i(x_i)$$

Firm i's profits π_i are

$$\pi_i \equiv x_i f(X) - C_i(x_i),\ \ i = 0,1,\cdots,n.$$

Under the Cournot behavioristic assumption, the first order conditions for the interior maximum of the public and private firm's outputs are as follows.

$$\frac{\partial W}{\partial x_0} = a - b(x_0 + X_{-0}) - C_0'(x_0) \equiv g(x_0, X_{-0}) = 0, \tag{7}$$

$$\frac{\partial \pi_i}{\partial x_i} = a - bX - bx_i - C_i'(x_i) \equiv h_i(x_i, X) = 0,\ \ i = 1,2,\cdots,n, \tag{8}$$

where $X_{-i} \equiv \sum_{j \neq i} x_j$.

Solving (7) with respect to x_0, we can get

$$\tilde{\phi}_0(X_{-0}) = \frac{1}{2c_2^0 + b}\left\{\left(a - c_1^0\right) - bX_{-0}\right\}.$$

Note that $\tilde{\phi}_0'(X_{-0}) = -\frac{b}{2c_2^0 + b} < 0$ and $\tilde{\phi}_0(\bar{X}^0) = 0$, where $\bar{X}^0 = \left(a - c_1^0\right)/b$ defined in (6). Since

$$g(0, X_{-0}) = a - bX_{-0} - C_0'(0) = a - bX_{-0} - c_1^0 < 0$$

for any $X_{-0} > \bar{X}^0$ and since $C_0''(x_0) \geq 0$ for any $x_0 \geq 0$

$$\begin{aligned} g(x_0, X_{-0}) &= a - b(x_0 + X_{-0}) - C_0'(x_0) \\ &\leq a - b(x_0 + X_{-0}) - C_0'(0) \\ &= g(0, X_{-0}) - bx_0 < 0 \end{aligned}$$

for any $x_0 \geq 0$ and $X_{-0} > \bar{X}^0$. This means that for any $X_{-0} > \bar{X}^0$, the public firm's choice of $x_0 = 0$ maximizes its profit. Hence, the best reply function of the public firm, $x_0 = \phi_0(X_{-0})$, has the following form.

$$\phi_0(X_{-0}) = \begin{cases} \frac{1}{2c_2^0 + b}\left\{\left(a - c_1^0\right) - bX_{-0}\right\} & \text{for} \quad X_{-0} \in \left[0, \bar{X}^0\right], \\ 0 & \text{for} \quad X_{-0} > \bar{X}^0. \end{cases} \tag{9}$$

As in Okuguchi and Yamazaki (2017), define

$$\hat{x}_0 = \phi_0(0) = \frac{a - c_1^0}{2c_2^0 + b}. \tag{10}$$

Next, we will derive each private firm's cumulative best reply function in the sense of Vives (Section 2.3.2, 1999) . Solving (8) with respect to x_i, we can get

$$\tilde{\Phi}_i(X) = \frac{1}{2c_2^i + b}\left\{\left(a - c_1^i\right) - bX\right\}.$$

Note that $\tilde{\Phi}_i'(X) = -\frac{b}{2c_2^i + b} < 0$ and $\tilde{\Phi}_i\left(\bar{X}^i\right) = 0$, where $\bar{X}^i = \left(a - c_1^i\right)/b$ defined in (6). Since

$$h_i(0, X) = a - bX - C_i'(0) = a - bX - c_1^i < 0$$

for any $X > \bar{X}^i$ and since $C_i''(x_i) \geq 0$ for any $x_i \geq 0$,

$$\begin{aligned} h_i(x_i, X) &= a - bX - bx_i - C_i'(x_i) \\ &\leq a - bX - bx_i - C_0'(0) \\ &= h_i(0, X) - bx_i < 0 \end{aligned}$$

for any $x_i \geq 0$ and $X > \bar{X}^i$. This means that for any $X > \bar{X}^i$, firm *i*'s choice of $x_i = 0$ maximizes its profit. Hence, the cumulative best reply function of firm *i*, $x_i = \Phi_i(X)$, has the following form.

$$\Phi_i(X) = \begin{cases} \tilde{\Phi}_i(X) = \frac{1}{2c_2^i + b}\left\{\left(a - c_1^i\right) - bX\right\} & \text{for} \quad X \in \left[0, \bar{X}^i\right], \\ 0 & \text{for} \quad X > \bar{X}^i. \end{cases} \tag{11}$$

Figure 1 shows the graph of $\Phi_i(X)$, $i = 1, 2$.

Define

$$\begin{aligned} \Phi_{-0}(X) &= \sum_{i=1}^{n} \Phi_i(X) \\ &= \begin{cases} \sum_{i \in I(X)} \frac{a - c_1^i}{2c_2^i + b} - X \sum_{i \in I(X)} \frac{b}{2c_2^i + b} & \text{for} \quad X \in \left[0, \bar{X}^{-0}\right], \\ 0 & \text{for} \quad X > \bar{X}^{-0}, \end{cases} \end{aligned} \tag{12}$$

where $I(X) \equiv \left\{i \middle| X \leq \bar{X}^i\right\}$ and

$$\bar{X}^{-0} \equiv \max\left\{\bar{X}^i\right\}_{i=1}^{n} = \bar{X}^1 = \frac{a - c_1^1}{b}. \tag{13}$$

Figure 1 shows in an example of $n = 2$ how $\Phi_{-0}(X)$ is constructed from $\Phi_i(X)$, $i = 1, 2$. Since $\Phi_i'(X) = \tilde{\Phi}_i'(X) < 0$ for any $X \in \left(0, \bar{X}^i\right)$ and $\Phi_i'(X) = 0$ for any $X > \bar{X}^i$, $\Phi_{-0}(X)$ is strictly decreasing in *X* for any $X \in \left[0, \bar{X}^{-0}\right)$, as in Figure 1.

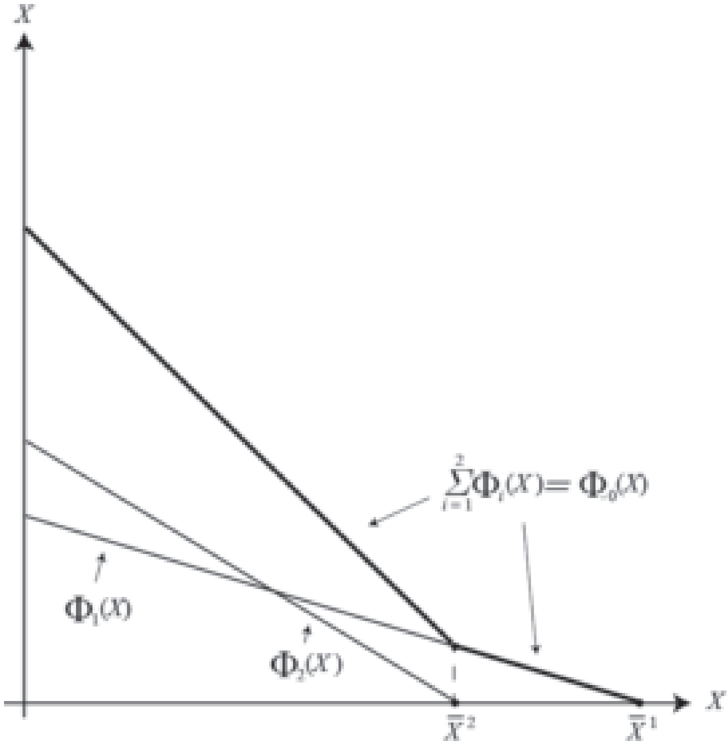

Figure 1 The graph of $\Phi_i(X)$ and $\Phi_{-0}(X)$, $i=1,2$

Denote the Nash equilibrium combination of (x, X_{-0}) as (x_0^*, X_{-0}^*). The equilibrium combination (x_0^*, X_{-0}^*) is the solution to the following two equations.

$$x_0 = \phi_0(X_{-0}), \tag{14}$$

$$X_{-0} = \Phi_{-0}(x_0 + X_{-0}). \tag{15}$$

Solving the equation (15) with respect to X_{-0}, we can get a new function

$$X_{-0} = \psi_{-0}(x_0), \tag{16}$$

where

$$\psi_{-0}(x_0) = \begin{cases} \dfrac{\sum_{i\in I(X)} \dfrac{a-c_1^i}{2c_2^i+b}}{1+\sum_{i\in I(X)} \dfrac{b}{2c_2^i+b}} - x_0 \dfrac{\sum_{i\in I(X)} \dfrac{b}{2c_2^i+b}}{1+\sum_{i\in I(X)} \dfrac{b}{2c_2^i+b}} & \text{for} \quad x_0 \in \left[0, \bar{X}^{-0}\right), \\ 0 & \text{for} \quad x_0 \geq \bar{X}^{-0}. \end{cases} \tag{17}$$

Define

$$\hat{X}_{-0} \equiv \psi_{-0}(0) = \frac{\sum_{i\in I(X)} \frac{a-c_1^i}{2c_2^i+b}}{1+\sum_{i\in I(X)} \frac{b}{2c_2^i+b}}. \tag{18}$$

Figure 2 shows how $X_{-0} = \psi_{-0}(x_0)$ is determined from the cumulative best reply function of the private firms, $X_{-0} = \Phi_{-0}(X)$.

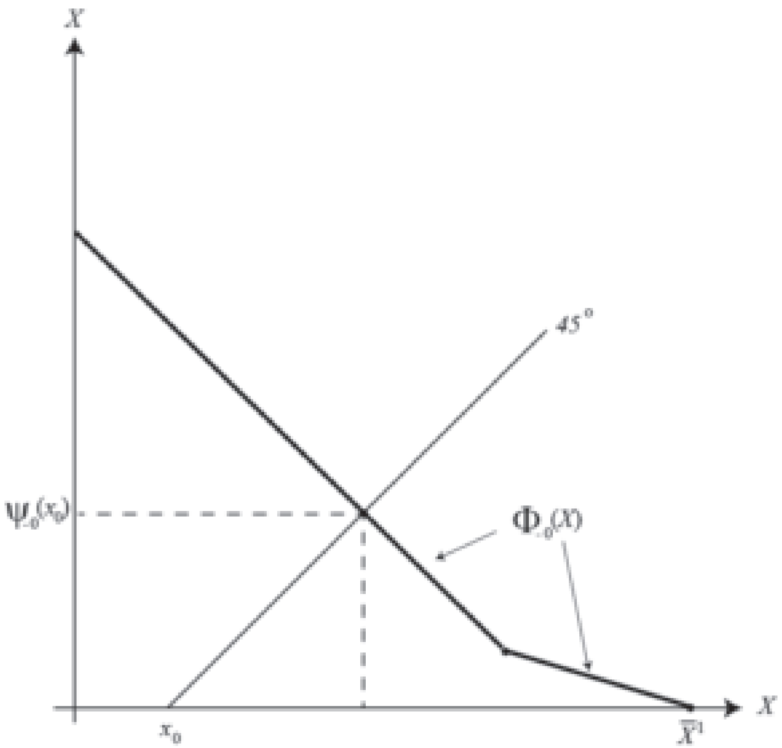

Figure 2 The graphical relation between $\Phi_{-0}(X)$ and $\psi_{-0}(x_0)$

Now we can restate that the Nash equilibrium combination $\left(x_0^*, X_{-0}^*\right)$ is the solution to two equations (14) and (16). As in Figure 3, the curve of $X_{-0} = \psi_{-0}(x_0)$ always intersects with that of $x_0 = \phi_0(X_{-0})$ only once.

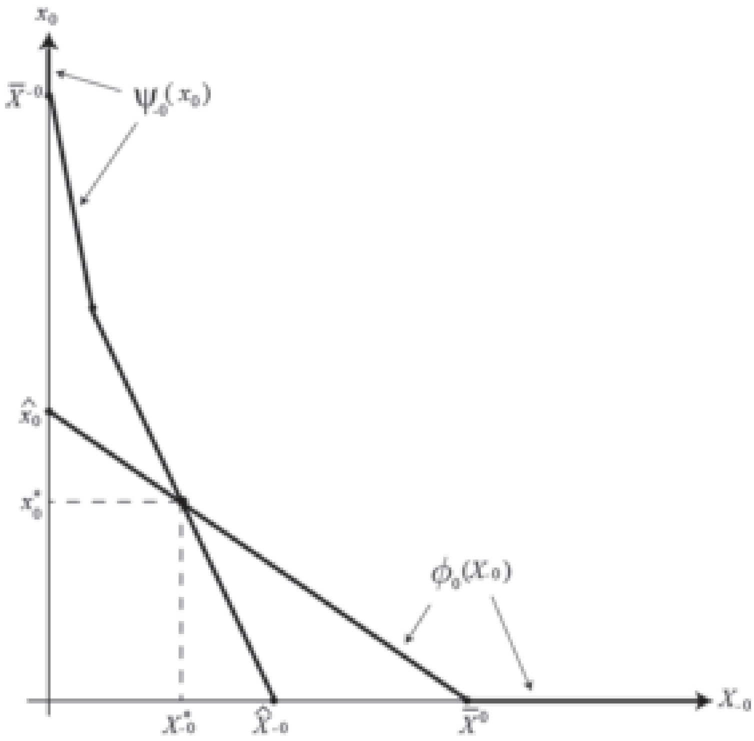

Figure 3 The Nash Equilibrium as a solution to two equations (14) and (16)

Hence, the following proposition holds.

Proposition 1: *There exists a unique Nash equilibrium in our model of Cournot mixed oligopoly with linear demand and quadratic cost functions.*

However, the public firm or all private firms may not produce the good in the unique equilibrium. Okuguchi and Yamazaki (2017) prove the following proposition.

Proposition 2: *Under Assumptions 1-6, if*

(C1) $$f\left(\hat{x}_0\right) = C_0'\left(\hat{x}_0\right) > \min\left\{C_i'\left(0\right)\right\}_{i=1}^{n},$$

(C2) $$f\left(\hat{X}_{-0}\right) > C_0'\left(0\right),$$

there exists a unique Nash equilibrium where the public firm and at least one private firm produce the good of the market.[4]

[4] Okuguchi (1985) assumes the condition (C2) only.

As Okuguchi and Yamazaki (2017) explain, since $f(\hat{x}_0) = C_0'(\hat{x}_0) > 0$ for some $i \neq 0$, if both $C_0'(0)$ and $\min\{C_i'(0)\}_{i=1}^n$ are small enough, the public firm and at least one private firm are active in the unique Nash equilibrium, provided that Assumptions 1-6 are satisfied. In our model, the conditions (C1) and (C2) are equivalent to

(C1’) $$\frac{a-c_1^0}{2c_2^0+b} = \hat{x}_0 < \overline{X}^{-0} = \frac{a-c_1^1}{b},$$

(C2’) $$\sum_{i\in I(X)} \frac{a-c_1^i}{2c_2^i+b}\left(1+\sum_{i\in I(X)}\frac{b}{2c_2^i+b}\right)^{-1} = \hat{X}_{-0} < \overline{X}^0 = \frac{a-c_1^0}{b},$$

respectively. It is easy to show that the inequality (C1’) is equivalent to

$$c_0^1 > \frac{2c_2^0+b}{b}c_1^1 - \frac{2c_2^0}{b}a. \tag{19}$$

Since $\sum_{i\in I(X)} \frac{a-c_1^1}{2c_2^i+b} \geq \sum_{i\in I(X)} \frac{a-c_1^i}{2c_2^i+b} \geq \sum_{i\in I(X)} \frac{a-c_1^n}{2c_2^i+b}$ by (4) and $\sum_{i\in I(X)} \frac{a-c_1}{2c_2^i+b}$ is strictly decreasing in c_1, there exists a unique positive number $\overline{c}_1 \in [c_1^1, c_1^n]$ such that

$$\sum_{i\in I(X)} \frac{a-\overline{c}_1}{2c_2^i+b} = \sum_{i\in I(X)} \frac{a-c_1^i}{2c_2^i+b}.$$

Roughly speaking, $\overline{c}_1$ is the “average” of $\{c_1^i\}_{i=1}^n$. Note that this $\overline{c}_1$ is a function of many exogenous variables including c_1^1. By the implicit function theorem, we can get

$$\frac{\partial \overline{c}_1}{\partial c_1^1} = \frac{1}{2c_2^1+b}\left(\sum_{i\in I(X)}\frac{1}{2c_2^i+b}\right)^{-1} < 1. \tag{20}$$

By using $\overline{c}_1$, we can rewrite (C2’) as

$$\sum_{i\in I(X)} \frac{a-\overline{c}_1}{2c_2^i+b}\left(1+\sum_{i\in I(X)}\frac{b}{2c_2^i+b}\right)^{-1} = \hat{X}_{-0} < \overline{X}^0 = \frac{a-c_1^0}{b}.$$

We can easily show that this inequality is equivalent to

$$c_1^0 < Aa - B(c_1^1)c_1^1, \tag{21}$$

where

$$A = \frac{\left(\sum_{i\in I(X)}\frac{b}{2c_2^i+b}\right)^{-1}}{1+\left(\sum_{i\in I(X)}\frac{b}{2c_2^i+b}\right)^{-1}} \in (0,1),\quad B(c_1^1) \equiv \frac{\alpha}{1+\left(\sum_{i\in I(X)}\frac{b}{2c_2^i+b}\right)^{-1}} > 0,\quad \alpha \equiv \overline{c}_1/c_1^1 \geq 1.$$

Because of $\overline{c}_1 \geq c_1^1$ and (20),

$$\frac{\partial \alpha}{\partial c_1^1} = \frac{1}{\left(c_1^1\right)^2}\left\{\frac{\partial \overline{c}_1}{\partial c_1^1} c_1^1 - \overline{c}_1\right\} < 0.$$

It is clear that $B\left(c_1^1\right)$ is strictly increasing in α. Hence, the above inequality proves that $B\left(c_1^1\right)$ is strictly decreasing in c_1^1.

The region of $\left(c_1^0, c_1^1\right)$ satisfying (19) and (21) is depicted in Figure 4. For the public firm and at least one of private firms to be active in the unique equilibrium, c_1^0 and c_1^1 cannot be larger than Aa and $\overline{c}_1^1$ determined in Figure 4, respectively. It may be interesting that c_1^1 which satisfies (19) and (21) for some c_1^0 has an upper bound and that the upper bound is maximized at a certain c_1^0 between 0 and Aa, i.e. $\hat{c}_1^0$ in Figure 4.

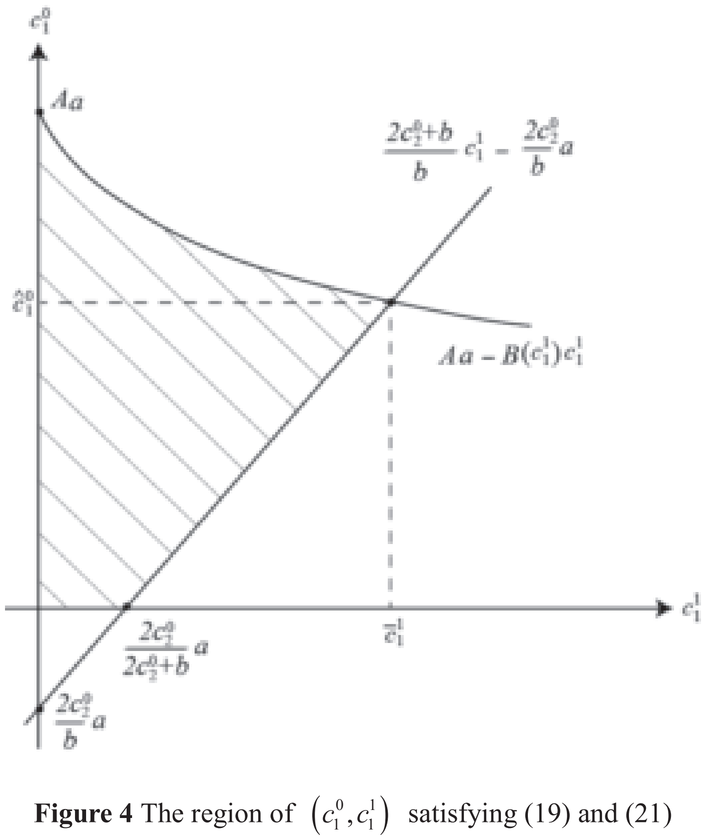

Figure 4 The region of $\left(c_1^0, c_1^1\right)$ satisfying (19) and (21)

4 Concluding Remarks

Okuguchi and Yamazaki (2017) prove that under the Assumptions 1-6, there exists a unique Nash equilibrium in a general model of Cournot mixed oligopoly with one public and one or more private firms. They also explain that under Assumptions 1-6, if each firm's marginal cost of its first unit of production is small enough, the public firm and at least one private firm produce the good in the unique Nash equilibrium. In this paper, we have shown how small each firm's marginal cost of its first unit of production should be in a concrete model of the mixed oligopoly where firms with a linear or quadratic cost function face a linear market demand function.

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